

# Numerical Solutions of Linear Fractional Nabla Difference Equations

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**Abstract:** In this article, we propose a novel matrix technique to obtain numerical solutions of initial value problems involving linear fractional nabla difference equations and provide few examples to illustrate the applicability of proposed method. In particular, we solve nabla analogue of fractional relaxation - oscillation equation numerically using this method.

**Key Words:** Fractional order, difference equation, triangular strip matrix, relaxation, oscillation

**AMS Classification:** 34A08, 39A23, 39A99.

## 1. INTRODUCTION

Discrete fractional calculus is a unified theory of arbitrary order sums and differences. Looking into the literature of fractional differences, we find two approaches: one using the  $\Delta$ -point of view (called the fractional delta difference), another using the  $\nabla$ -perspective (called the fractional nabla difference). In this article, we confine ourselves to the second approach.

The concept of fractional nabla difference traces back to the work of Gray & Zhang [10]. They introduced a new definition of fractional nabla difference which overcame all drawbacks of the earlier definitions given by Granger and Joyeux [9] and Hosking [12]. Later, Anastassiou [8] and Atici & Elloe [5] slightly modified Gray's definition. In the past one decade, there has been a lot of research on this topic [2, 5, 6, 7, 8, 13, 14, 15, 16, 17, 19, 20, 23, 24].

Unlike the integer case, solving a linear fractional nabla difference equation with constant co-efficients (LFNDE) is quite complicated. Atici & Elloe [7, 20], Hein et al. [11] and Jaganmohan Jonnalagadda [14] developed N-transform method to find the analytical solutions of initial value problems involving LFNDEs. Fahd Jarad et al. [4] applied the generalized nabla discrete Sumudu transform to solve LFNDEs with initial value problems. Atici & Elloe [7] constructed the fundamental matrix for a homogeneous system of LFNDEs and the causal Green's function for a nonhomogeneous system of LFNDEs. Acar & Atici [20] defined the characteristic equation of a sequential LFNDE and obtained the general solution in two cases where the characteristic real roots are same or distinct. Atici & Elloe [6] obtained the equivalence of an initial value problem for a LFNDE and a linear fractional sum equation and gave an explicit solution to the linear fractional sum equation. Jia et al. [18] established comparison theorems and asymptotic results for initial value problems associated with LFNDEs.

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The present article is organized as follows: Section 2 contains preliminaries on fractional nabla calculus. In section 3, we transform a LFNDE together with initial conditions into a system of linear equations and obtain the numerical solution. Few examples are provided in section 4 to demonstrate the applicability of established results.

## 2. PRELIMINARIES

Throughout this article, we use the following notations, definitions and known results of fractional nabla calculus: Denote the set of all real numbers by  $\mathbb{R}$ . Define  $\mathbb{N}_a = \{a, a+1, a+2, \dots\}$  and  $\mathbb{N}_a^b = \{a, a+1, a+2, \dots, b\}$  for any  $a, b \in \mathbb{R}$  such that  $a < b$ .

**Definition 2.1.** (Rising Factorial Function) For any  $\alpha \in \mathbb{R}$  and  $t \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$  such that  $(t + \alpha) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$ , the rising factorial function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t + \alpha)}{\Gamma(t)}, \quad 0^{\overline{\alpha}} = 0.$$

**Definition 2.2.** (Taylor Monomial) For any  $t \in \mathbb{R}$  and  $\alpha \in \mathbb{R} \setminus \{\dots, -2, -1\}$ , the  $\alpha^{\text{th}}$ -Taylor monomial is defined by

$$h_\alpha(t, a) = \frac{(t - a)^{\overline{\alpha}}}{\Gamma(\alpha + 1)}.$$

**Lemma 2.1.** We observe the following properties of Taylor monomials.

- (1)  $h_\alpha(t + 1, t) = 1$ .
- (2)  $h_\alpha(t, a) = h_\alpha(t - a, 0)$ .
- (3)  $h_\alpha(t + 1, a) - h_{\alpha-1}(t + 1, a) = h_\alpha(t, a)$ .

**Definition 2.3.** Let  $\alpha \in \mathbb{R}$  and choose  $N \in \mathbb{N}_1$  such that  $N - 1 < \alpha < N$ .

- (1) (Fractional Nabla Sum) [5] The  $\alpha^{\text{th}}$ -order nabla sum of  $u$  is given by

$$(\nabla_a^{-\alpha} u)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^t (t - s + 1)^{\overline{\alpha-1}} u(s), \quad t \in \mathbb{N}_a.$$

- (2) (R-L Fractional Nabla Difference) [5] The  $\alpha^{\text{th}}$ -order nabla difference of  $u$  is given by

$$(\nabla_a^\alpha u)(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^t (t - s + 1)^{\overline{-\alpha-1}} u(s), \quad t \in \mathbb{N}_{a+N}.$$

- (3) (Caputo Fractional Nabla Difference) [23] The  $\alpha^{\text{th}}$ -order nabla difference of  $u$  is given by

$$(\nabla_{a*}^\alpha u)(t) = (\nabla_a^\alpha u)(t) - \sum_{k=0}^{N-1} \frac{(t - a - k + 1)^{\overline{k-\alpha}}}{\Gamma(k - \alpha + 1)} (\nabla_a^k u)(a + k), \quad t \in \mathbb{N}_{a+N}.$$

## 3. METHOD OF SOLUTION

Using triangular strip matrices, Podlubny [22] described a matrix approach to find numerical solutions of fractional differential equations. Motivated by this technique, in this section, we present a matrix method to solve linear fractional nabla difference equations.

Let  $m \in \mathbb{N}_1$  and  $a, f : \mathbb{N}_0^m \rightarrow \mathbb{R}$  such that  $a(t) \neq -1$  for all  $t \in \mathbb{N}_0^m$ . Consider a linear fractional nabla difference equation of Riemann-Liouville type

$$(\nabla_0^\alpha u)(t) + a(t)u(t) = f(t), \quad 0 < \alpha < 1, \quad t \in \mathbb{N}_1^m. \quad (3.1)$$

Take  $u(0) = c$ ,  $\tilde{u} = [u(1), u(2), \dots, u(m)]^T$  and  $\mathcal{F} = [f(1), f(2), \dots, f(m)]^T$ . Then, the matrix form of (3.1) is

$$\mathcal{L}\tilde{u} = \mathcal{F} - c\mathcal{B},$$

where

$$\mathcal{L} = \begin{pmatrix} 1 + a(1) & 0 & \cdots & \cdots & 0 & 0 \\ h_{-\alpha-1}(2, 0) & 1 + a(2) & \cdots & \cdots & 0 & 0 \\ h_{-\alpha-1}(3, 0) & h_{-\alpha-1}(2, 0) & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h_{-\alpha-1}(m-1, 0) & h_{-\alpha-1}(m-2, 0) & \cdots & \cdots & 1 + a(m-1) & 0 \\ h_{-\alpha-1}(m, 0) & h_{-\alpha-1}(m-1, 0) & \cdots & \cdots & h_{-\alpha-1}(2, 0) & 1 + a(m) \end{pmatrix}_{m \times m}$$

is a lower triangular strip matrix and

$$\mathcal{B} = \begin{pmatrix} h_{-\alpha-1}(2, 0) \\ h_{-\alpha-1}(3, 0) \\ h_{-\alpha-1}(4, 0) \\ \vdots \\ \vdots \\ h_{-\alpha-1}(m, 0) \\ h_{-\alpha-1}(m+1, 0) \end{pmatrix}_{m \times 1}.$$

Since  $\mathcal{L}$  is non-singular,  $\tilde{u} = \mathcal{L}^{-1}[\mathcal{F} - c\mathcal{B}]$ .

Replacing the Riemann-Liouville type difference operator in (3.1) with the Caputo operator, the matrix form of

$$(\nabla_{0*}^\alpha u)(t) + a(t)u(t) = f(t), \quad 0 < \alpha < 1, \quad t \in \mathbb{N}_1^m, \quad (3.2)$$

is

$$\mathcal{L}\tilde{u} = \mathcal{F} + c\mathcal{C},$$

where

$$\mathcal{C} = \begin{pmatrix} h_{-\alpha}(1, 0) \\ h_{-\alpha}(2, 0) \\ h_{-\alpha}(3, 0) \\ \vdots \\ \vdots \\ h_{-\alpha}(m-1, 0) \\ h_{-\alpha}(m, 0) \end{pmatrix}_{m \times 1}.$$

Since  $\mathcal{L}$  is non-singular,  $\tilde{u} = \mathcal{L}^{-1}[\mathcal{F} + c\mathcal{C}]$ .

Let  $n \in \mathbb{N}_2$  and  $a, b, f : \mathbb{N}_0^n \rightarrow \mathbb{R}$  such that  $a(t) + b(t) \neq -1$  for all  $t \in \mathbb{N}_0^n$ . Now, consider a sequential multi-term fractional nabla difference equation of Riemann-Liouville type

$$(\nabla_0^\beta u)(t) + a(t)(\nabla_0^\alpha u)(t) + b(t)u(t) = f(t), \quad 0 < \alpha < \beta < 2, \quad t \in \mathbb{N}_2^n. \quad (3.3)$$

Take  $u(0) = c$ ,  $u(1) = d$ ,  $\tilde{u} = [u(2), u(3), \dots, u(n)]^T$  and  $\mathcal{F} = [f(2), f(3), \dots, f(n)]^T$ . Then, the matrix form of (3.3) is

$$\mathcal{L}\tilde{u} = \mathcal{F} - c\mathcal{P} - d\mathcal{Q}.$$

Here

$$\mathcal{L} = [\mathcal{L}_{ij}]_{(n-1) \times (n-1)}, \text{ where}$$

$$\mathcal{L}_{ij} = 0, 1 \leq i < j \leq (n-1),$$

$$\mathcal{L}_{ii} = 1 + a(i+1) + b(i+1), 1 \leq i \leq (n-1),$$

$$\mathcal{L}_{ij} = h_{-\beta-1}(i-j+1, 0) + a(i+1)h_{-\alpha-1}(i-j+1, 0), 1 \leq j < i \leq (n-1),$$

is a lower triangular strip matrix and

$$\mathcal{P} = \begin{pmatrix} h_{-\alpha-1}(3, 0) \\ h_{-\alpha-1}(4, 0) \\ h_{-\alpha-1}(5, 0) \\ \vdots \\ \vdots \\ h_{-\alpha-1}(n, 0) \\ h_{-\alpha-1}(n+1, 0) \end{pmatrix}_{(n-1) \times 1} \quad \text{and} \quad \mathcal{Q} = \begin{pmatrix} a(2)h_{-\alpha-1}(3, 0) \\ a(3)h_{-\alpha-1}(4, 0) \\ a(4)h_{-\alpha-1}(5, 0) \\ \vdots \\ \vdots \\ a(n-1)h_{-\alpha-1}(n, 0) \\ a(n)h_{-\alpha-1}(n+1, 0) \end{pmatrix}_{(n-1) \times 1}.$$

Since  $\mathcal{L}$  is non-singular,  $\tilde{u} = \mathcal{L}^{-1}[\mathcal{F} - c\mathcal{P} - d\mathcal{Q}]$ .

Replacing the Riemann-Liouville type difference operator in (3.3) with the Caputo operator, we have

$$(\nabla_{0*}^\beta u)(t) + a(t)(\nabla_{0*}^\alpha u)(t) + b(t)u(t) = f(t), \quad 0 < \alpha < \beta < 2, \quad t \in \mathbb{N}_2^n. \quad (3.4)$$

An application of definition 2.3 reduces the above equation to a fractional nabla difference equation of Riemann-Liouville type

$$(\nabla_0^\beta u)(t) + a(t)(\nabla_0^\alpha u)(t) + b(t)u(t) = g(t), \quad 0 < \alpha < \beta < 2, \quad t \in \mathbb{N}_2^n, \quad (3.5)$$

where  $g : \mathbb{N}_0^n \rightarrow \mathbb{R}$ .

#### 4. EXAMPLES

We can compute nabla exponential functions using this technique. Consider the Riemann-Liouville type eigenvalue problem

$$(\nabla_0^\alpha u)(t) + \lambda u(t) = 0, \quad -1 < \lambda < 1, \quad 0 < \alpha < 1, \quad t \in \mathbb{N}_1^m. \quad (4.1)$$

Let  $u(0) = 1$ . Then, the solution of (4.1) is

$$u(t) = \hat{e}_{\alpha, \alpha}(-\lambda, t^{\bar{\alpha}}), \quad t \in \mathbb{N}_0^m. \quad (4.2)$$

The matrix form of (4.1) is

$$\mathcal{L}\tilde{u} = -\mathcal{B}.$$

Since  $\mathcal{L}$  is non-singular,  $\tilde{u} = -\mathcal{L}^{-1}\mathcal{B}$ . Then, from (4.2), we have

$$\tilde{u} = \hat{e}_{\alpha, \alpha}(-\lambda, t^{\bar{\alpha}}), \quad t \in \mathbb{N}_0^m.$$

The graphs of  $\hat{e}_{0.5,0.5}(-0.5, t^{\overline{0.5}})$  and  $\hat{e}_{0.5,0.5}(0.5, t^{\overline{0.5}})$  for  $t \in \mathbb{N}_1^{100}$  are shown in Figures 1 and 2 respectively.

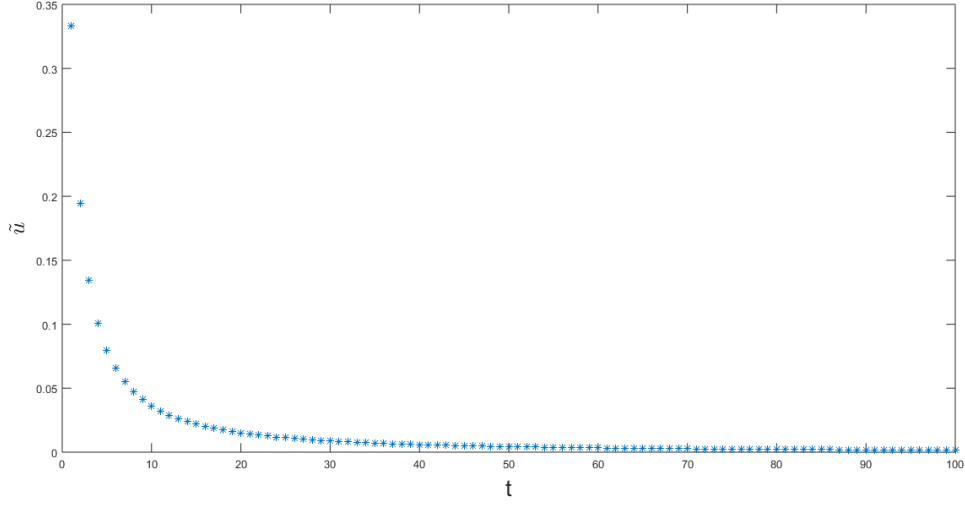


FIGURE 1.

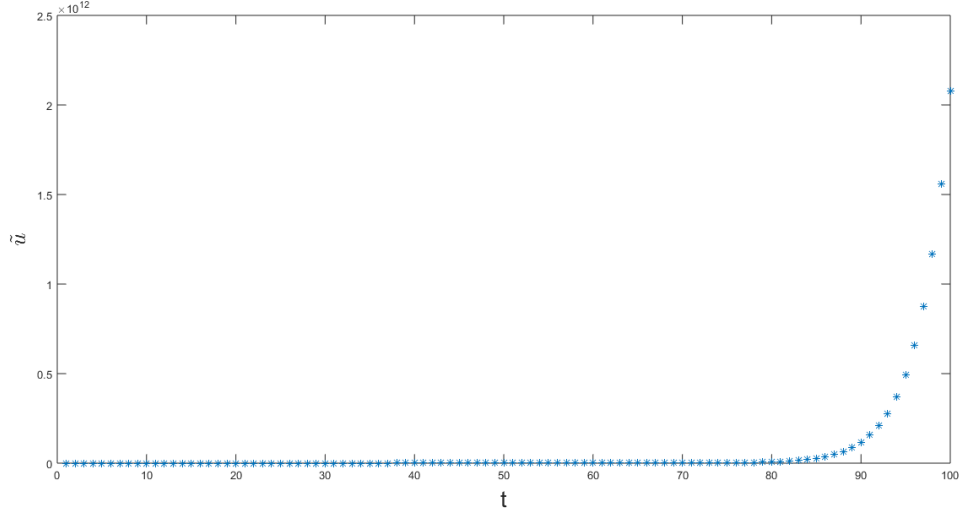


FIGURE 2.

Now consider the Caputo type eigenvalue problem

$$(\nabla_{0*}^\alpha u)(t) + \lambda u(t) = 0, \quad -1 < \lambda < 1, \quad 0 < \alpha < 1, \quad t \in \mathbb{N}_1^m. \quad (4.3)$$

Let  $u(0) = 1$ . Then, the solution of (4.3) is

$$u(t) = \hat{e}_\alpha(-\lambda, t^{\overline{\alpha}}), \quad t \in \mathbb{N}_0^m. \quad (4.4)$$

The matrix form of (4.3) is

$$\mathcal{L}\tilde{u} = C.$$

Since  $\mathcal{L}$  is non-singular,  $\tilde{u} = \mathcal{L}^{-1}C$ . Then, from (4.4), we have

$$\tilde{u} = \hat{e}_\alpha(-\lambda, t^{\overline{\alpha}}), \quad t \in \mathbb{N}_0^m.$$

The graphs of  $\hat{e}_{0.5}(0.5, t^{\overline{0.5}})$  and  $\hat{e}_{0.5}(-0.5, t^{\overline{0.5}})$  for  $t \in \mathbb{N}_1^{100}$  are shown in Figures 3 and 4 respectively.

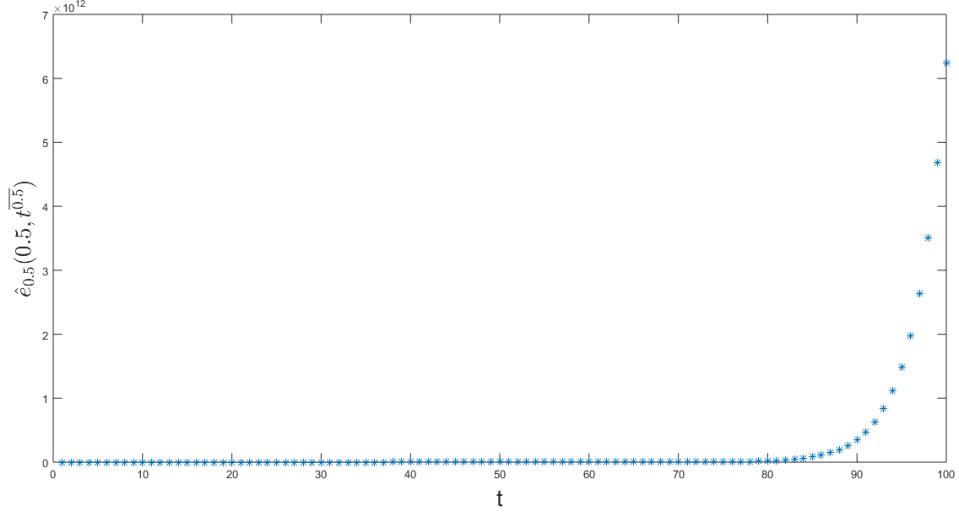


FIGURE 3.

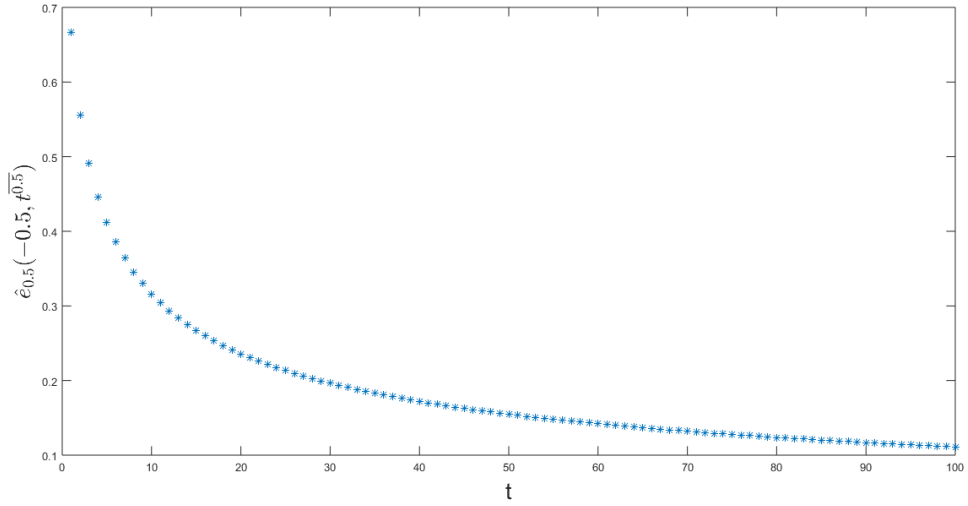


FIGURE 4.

**Example 1.** *The solution of*

$$(\nabla_0^{0.5}u)(t) = \frac{1}{2^{t+1}}u(t), \quad t \in \mathbb{N}_1^{100}, \quad (4.5)$$

$$u(0) = 1, \quad (4.6)$$

*is shown in Figure 5.*

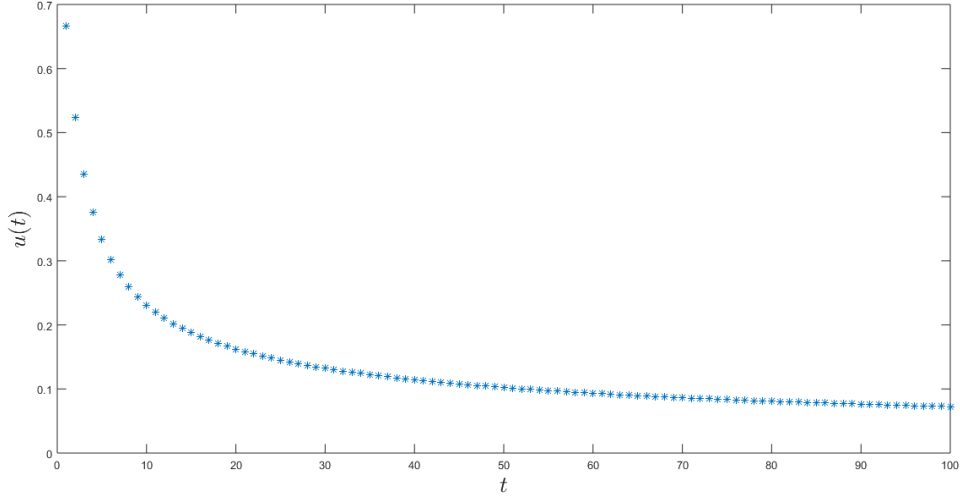


FIGURE 5.

**Example 2.** *The solution of*

$$(\nabla_0^{0.5}u)(t) = \frac{1}{2^{t+1}}u(t) + \frac{1}{2(t+1)}, \quad t \in \mathbb{N}_1^{100}, \quad (4.7)$$

$$u(0) = 1, \quad (4.8)$$

*is shown in Figure 6.*

## 5. APPLICATION

In this section, we apply the proposed method to solve fractional nabla difference analogue of relaxation - oscillation equation.

**5.1. Relaxation Equation.** The discrete fractional nabla relaxation model can be depicted as

$$(\nabla_{0*}^\alpha u)(t) + Au(t) = f(t), \quad 0 < \alpha < 1, \quad t \in \mathbb{N}_1^m, \quad (5.1)$$

$$u(0) = c. \quad (5.2)$$

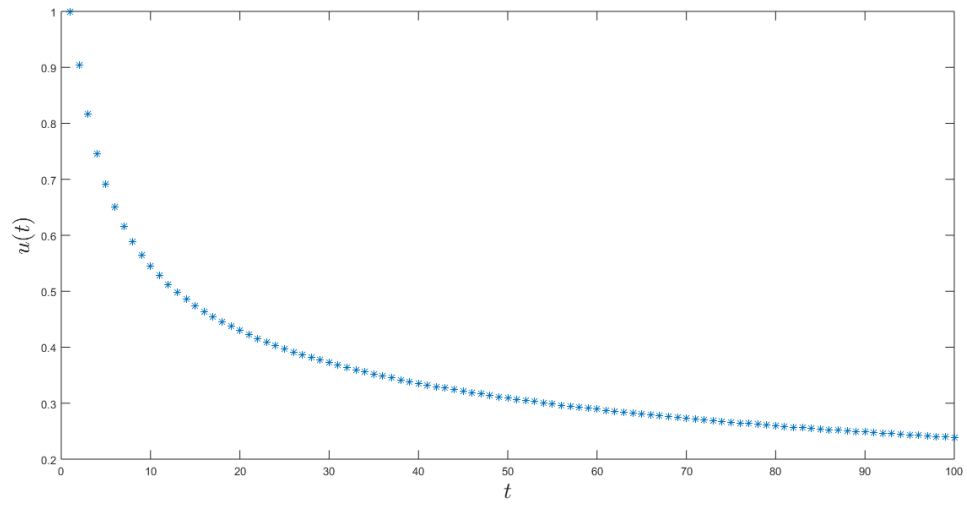


FIGURE 6.

**Example 3.** *The solution of*

$$(\nabla_{0*}^{0.5}u)(t) + (0.5)u(t) = \frac{1}{2^{t+1}} - \frac{(t+1)^{-0.5}}{\Gamma(0.5)}, \quad t \in \mathbb{N}_1^{100}, \quad (5.3)$$

$$u(0) = 1, \quad (5.4)$$

*is shown in Figure 7.*

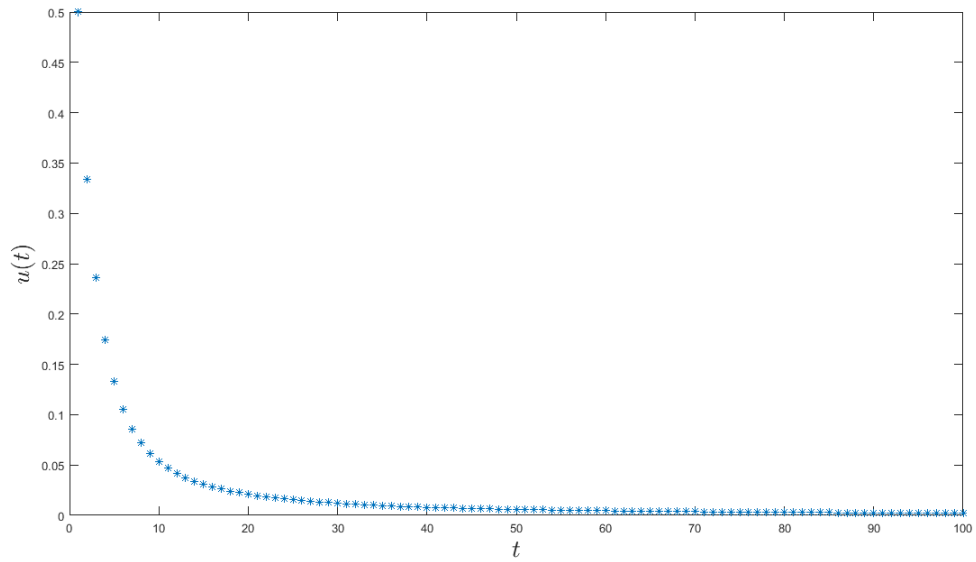


FIGURE 7.



**5.2. Oscillation Equation.** The discrete fractional nabla oscillation model can be described as

$$(\nabla_{0*}^{\beta}u)(t) + Bu(t) = f(t), \quad 0 < \beta < 2, \quad t \in \mathbb{N}_2^n, \quad (5.5)$$

$$u(0) = c, \quad u(1) = d. \quad (5.6)$$

**Example 4.** The solution of

$$(\nabla_{0*}^{1.5}u)(t) + (0.5)u(t) = \frac{1}{2^{t+1}} - \frac{(t+1)^{-1.5}}{\Gamma(-0.5)}, \quad t \in \mathbb{N}_1^{100}, \quad (5.7)$$

$$u(0) = 1, \quad u(1) = 1, \quad (5.8)$$

is shown in Figure 8.

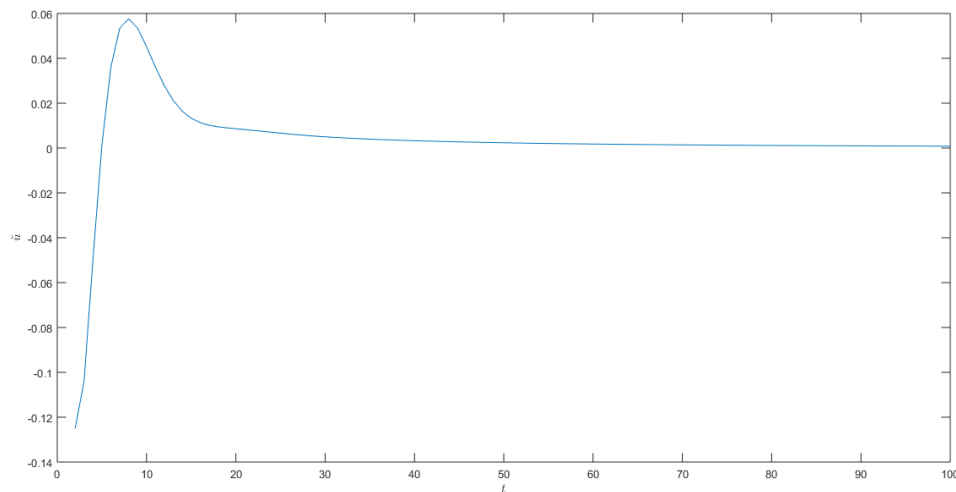


FIGURE 8.

#### CONFLICT OF INTEREST

The author declares that there is no conflict of interest regarding the publication of this article.

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